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DISPERSION POINTS AND CONTINUOUS FUNCTIONS

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The aim of this paper is to answer the following question raised by J. Cobb and W. Voxman in 1980:

If X is a connected space with a dispersion point p, and if $f: X \to X$ is a nonconstant continuous function, then is $f(p)=p$?

The answer to this question is negative, and we give a counterexample along with three theorems on a space with a dispersion point.

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dispersation points fixed points quasi-components

In this paper, a topological space always means a T_1 space, i.e., a singleton set is closed, unless otherwise stated. A point p in a connected space X with more than one element is said to be a *dispersion point* of X if each component of $X-\{p\}$ consists of a single element, i.e., if $X-\{p\}$ is totally disconnected. We shall call such a space a *dispersion point space*. It is well known that no space can have more than one dispersion point. (If the space is not T_1, this assertion is not true. If we let $X=\{x, y\}$ with only three open sets $\emptyset$, $\{x\}$, and X, then X is a connected space, and both x and y are dispersion points.) The best known example of a dispersion point space is the following example due to Knaster and Kuratowski [4].

Example 1. Let $C=\{\sum_{n=1}^{\infty} a_n/3^n$: for each $n=1,2,\ldots, a_n=0$ or $2\}$, the Cantor ternary set. Let p be the point $(\frac{1}{2}, \frac{1}{2})$ in the plane. For each z in C, the line segment joining p and $(z, 0)$ on the x-axis is denoted by l_z. For each z in C, let $\hat{l}_z$ be the set of points (x, y) in l_z such that

(1) if z is rational, then so are x and y, and
(2) if z is irrational, then so are x and y.

Let $X=\bigcup_{z\in C} \hat{l}_z$. Then X is a connected space (with the relative topology inherited from the plane) with the dispersion point p. This space is called the *Cantor teepee* (see Fig. 1).

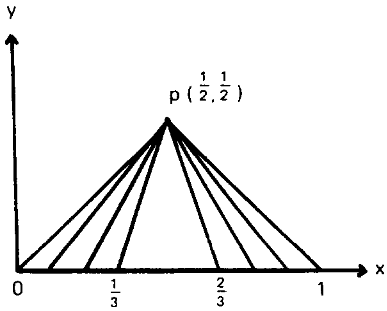

Fig. 1

For other interesting examples of dispersion point spaces, see [2], [3], [5], [6], [7], and [8].

If f is a mapping (mapping always means a continuous function) from a space X into itself, then a point x in X is said to be a *fixed point* of f if $f(x)=x$. If each mapping from X into itself has a fixed point, then X is said to have the *fixed point property*. In 1980, Cobb and Voxman wrote an article [1] on dispersion point spaces, and observed that if X is any one of the above examples with a dispersion point p, and if f is a nonconstant mapping on X into itself, then $f(p)=p$. Consequently, they raised the following question.

Question 1. Suppose that X is a connected space with a dispersion point p, and f a nonconstant mapping on X into itself. Then is $f(p)=p$?

In this note, we will prove two theorems related to the above question, a theorem that will enable us to construct many dispersion point spaces, and finally we will construct a counterexample to the conjecture.

Theorem 1. *Let X be a connected space with a dispersion point p. Suppose, for each open neighborhood U of p in X, the component of U that contains p is open in X. Then, for each nonconstant mapping f from X into itself, we must have $f(p)=p$.*

Proof. Suppose there exists a nonconstant mapping $f: X\to X$ such that $f(p)\neq p$. Since $f(X)$ is a nondegenerate connected subspace of X, p must be in $f(X)$. So $f^{-1}(p)\neq\emptyset$, and p is in $X-f^{-1}(p)$. Let K be the component of the open set $X-f^{-1}(p)$ that contains p. By our hypothesis, K is an open subset of X.

On the other hand, since $f(K)$ is connected and $f(K)$ is contained in $X-\{p\}$, $f(K)$ must be equal to $\{f(p)\}$. Hence

$$p\in K\subset f^{-1}f(p)\subset X-f^{-1}(p).$$

This implies that K is a component of the closed subset $f^{-1}f(p)$. Therefore, K is a proper closed and open subset of X. But this is a contradiction to the connectedness of the space X. □

Theorem 2. *Let X be a connected space with a dispersion point p. Suppose that, for each open neighborhood U of p, the closure of the component K of U that contains p meets the boundary of U, i.e.,*

$$\bar{K} \cap \mathrm{Fr}(U) \neq \emptyset.$$

Then, for each nonconstant mapping $f: X \to X$, we must have $f(p) = p$.

Proof. Suppose that $f(p) \neq p$ for some nonconstant mapping f on X. Then p is in $f^{-1}(X - \{p\})$ and $f^{-1}(X - \{p\})$ is open in X. Let K be the component of $f^{-1}(X - \{p\})$ that contains p. Since $f(K)$ is a connected subset of the totally disconnected space $X - \{p\}$, $f(K)$ is a singleton set, namely $\{f(p)\}$. But

$$\emptyset \neq \bar{K} \cap \mathrm{Fr}(f^{-1}(X - \{p\}) \subset \bar{K} \cap (X - f^{-1}(X - \{p\})) \subset f^{-1}(p).$$

Hence, we must have

$$p \in f(\bar{K}) \subset \overline{f(K)} = \{f(p)\}.$$

But this is a contradiction to $f(p) \neq p$. $\square$

We can see that either Theorem 1 or Theorem 2 can be applied to the Cantor teepee to conclude that the dispersion point is a fixed point of a nonconstant mapping. Moreover, these theorems simplify the proof of Theorem 1 in [1].

Definition. A *quasi-component* of a space X that contains the element x of X is the intersection of all closed-open sets containing x.

Notation. If X is not a connected space, then "$X = M \cup N$, a separation" means that M and N are disjoint nonempty closed subset of X whose union is X.

Notation. If F is nonempty closed subset of a space X, then X/F is used to denote the quotient space obtained from X by identifying F to a single point.

The next theorem enable us to construct infinitely many dispersion point spaces.

Theorem 3. *Suppose X is a totally disconnected space, and $\{Y_a$: a is in $A\}$ the collection of all quasi-components of X. Suppose F is a proper nonempty closed subset of X that meets Y_a for each a in A. Let $q: X \to X/F$ be the quotient mapping. Then X/F is a connected space with the dispersion point $q(F)$.*

Proof. Since $X/F - \{q(F)\}$ is homeomorphic to $X - F$, $X/F - \{q(F)\}$ is totally disconnected. Suppose X/F is not connected, say $X/F = M \cup N$, a separation. We assume that $q(F)$ is in M. Then $q^{-1}(N)$ is a closed-open subset of X. Thus, for some a in A, Y_a is contained in $q^{-1}(N)$. But $q(F)$ is in M so that $f \cap q^{-1}(N) = \emptyset$. This implies that $F \cap Y_a = \emptyset$, which is a contradiction. □

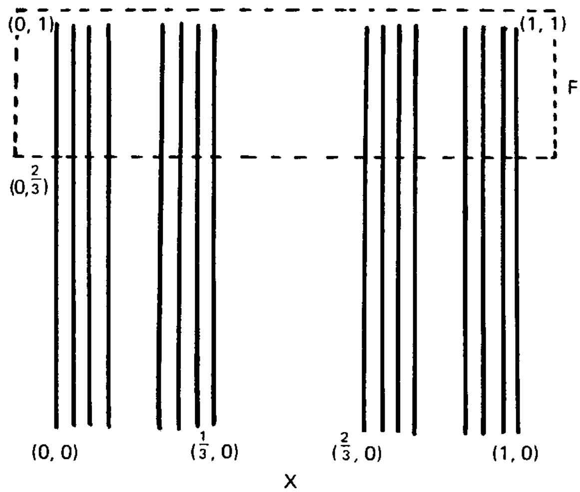

Fig. 2

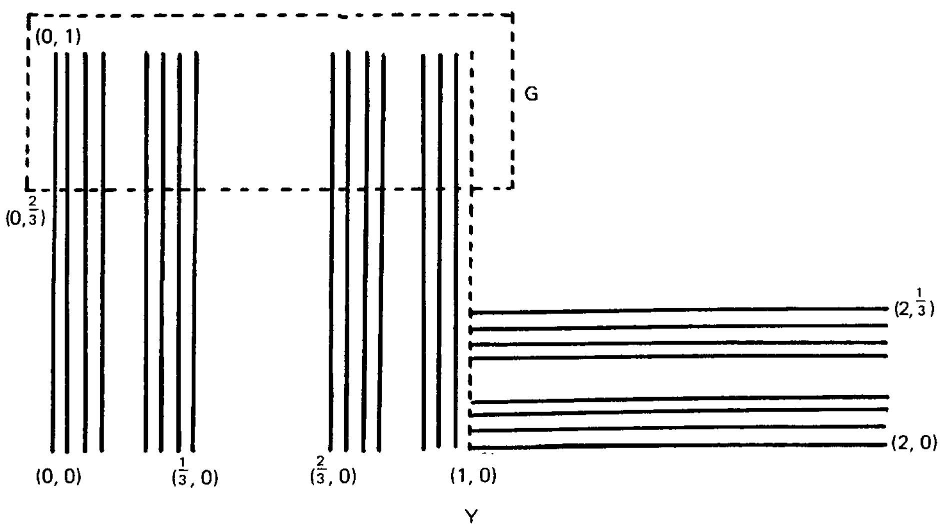

Fig. 3

Examples 3. Again, let C be the Cantor ternary set. For each x in C, let

$$l_x = \left\{ (x, y) \,\middle|\, \begin{array}{l} 0 \leqslant y \leqslant 1, \\ \text{if } x \text{ is rational, so is } y, \text{ and} \\ \text{if } x \text{ is irrational, then so is } y. \end{array} \right\}.$$

(a) Let $X = \bigcup_{x \in C} l_x$, and $F = \{(x, y) \in X : \frac{2}{3} \leqslant y \leqslant 1\}$. Then X is totally disconnected, and $\{l_x : x \text{ is in } C\}$ is the collection of all quasi-components of X. Moreover, F is a proper closed subset of X that meets l_x for each x in C. Hence, X/F is a dispersion point space by Theorem 3. Note that X/F is homeomorphic to the Cantor teepee (see Fig. 2).

(b) Let X and F be the same as in (a). Let $E = F \cup \{(0, 0)\}$. Then X/E is also a dispersion point space by Theorem 3. Note that Theorem 1 does not apply to X/E, while Theorem 2 does apply to X/E to conclude that the dispersion point is a fixed point of a nonconstant mappings.

(c) The next example shows that not all dispersion point spaces are the type of space described in Theorem 3.

For every y in C, let

$$L_y = \left\{ (x, \tfrac{1}{3}y) \,\middle|\, \begin{array}{l} 1 \leqslant x \leqslant 2, \\ \text{if } y \text{ is rational, so is } x, \text{ and} \\ \text{if } y \text{ is irrational, so is } x. \end{array} \right\}.$$

(See Fig. 3.) Let $L = \bigcup_{y \in C} L_y$. We define a space Y to be $[\bigcup \{l_x : x \in C - \{1\}\}] \cup L$. Then Y is a totally disconnected space, and $\{l_x : x \in C - \{1\}\} \cup \{L\}$ is the collection of all quasi-components of Y. Let G be the set $\{(x, y) \in Y : \frac{2}{3} \leqslant y \leqslant 1\}$. Then G is a closed subset of Y that does not meet L. Yet, Y/G is a dispersion point space.

Let $q : Y \to Y/G$ be the quotient map. It is easy to see that $Y/G - \{q(G)\}$ is totally disconnected. Suppose that $Y/G = M \cup N$, a separation. Say $q(G)$ is in M. Then $G \cap q^{-1}(N) = \emptyset$ and $q^{-1}(N)$ is closed-open. So L must be contained in $q^{-1}(N)$. But then, for some x in $C - \{l\}$, $l_x \cap q^{-1}(N) \neq \emptyset$. This implies that l_x is contained in $q^{-1}(N)$ and, therefore, $G \cap q^{-1}(N) \neq \emptyset$. This is a contradiction. Hence Y/G is a connected space with a dispersion point $q(G)$. By Theorem 2, $q(G)$ is a fixed point of a nonconstant mapping.

Finally, we are ready to present a counter-example to Question 1.

Example 4. As above, C is the Cantor ternary set. For each y in C, we define A_y as a union of the following five sets:

$$\left\{ (x, y, 2) \,\middle|\, \begin{array}{l} x \in \bigcup_{n=0}^{\infty} \left[1 - \frac{1}{2^{2n}}, 1 - \frac{1}{2^{2n+1}} \right], \\ \text{if } y \text{ is rational, so is } x, \text{ and} \\ \text{if } y \text{ is irrational, then so is } x \end{array} \right\},$$

$$\left\{(x, y, 0)\;\middle|\; \begin{array}{l} x \in \bigcup_{n=1}^{\infty} \left[1-\frac{1}{2^{2n-1}}, 1-\frac{1}{2^{2n}}\right], \\ \text{if } y \text{ is rational, so is } x, \text{ and} \\ \text{otherwise, } x \text{ is irrational} \end{array}\right\},$$

$$\left[\bigcup_{n=0}^{\infty} \left\{\left(1-\frac{1}{2^n}, y, z\right) \;\middle|\; \begin{array}{l} z \in [0, 2], \\ \text{if } y \text{ is rational, so is } z, \\ \text{otherwise, } z \text{ is irrational} \end{array}\right\}\right],$$

$$\left\{(1, y, z) \;\middle|\; \begin{array}{l} z \in [0, 1], \\ \text{if } y \text{ is rational, so is } z, \\ \text{otherwise, } z \text{ is irrational} \end{array}\right\},$$

and the set $\{(1, y, 0)\}$. (See Figs. 4 and 5.)

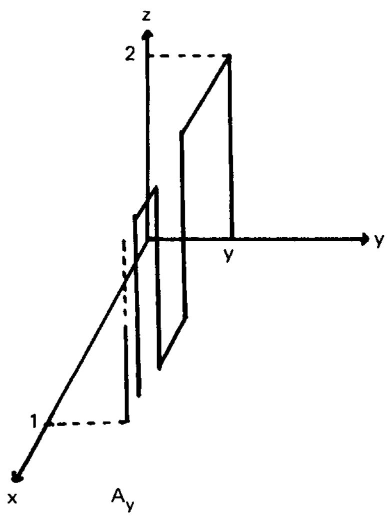

Fig. 4

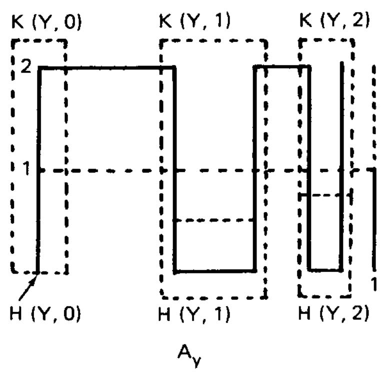

Fig. 5

Let $X = \bigcup\{A_y : y \in C\}$. Then X is totally disconnected space with the usual topology, and $\{A_y : y \in C\}$ is the collection of all quasi-components of X. Let $F = \{(1, y, 0) : y \in C\}$. Then F is a proper closed subset of X that meets A_y for each y in C. Let q be the quotient mapping from X onto X/F. Then, by Theorem 3, X/F is a connected space with a dispersion point $q(F)$. (See Fig. 6.) Now, for each y in C and for each $n = 0, 1, 2, \ldots$, we define $H(y, n)$ and $K(y, n)$ as follows (see Fig. 5):

$$H(y, n) = \left\{ (x, y, z) \in A_y \;\middle|\; \begin{array}{l} x = 0 \text{ if } n = 0, \\ x \in \left[1 - \dfrac{1}{2^{2n-1}}, 1 - \dfrac{1}{2^{2n}}\right] \text{ if } n \neq 0, \\ z \in \left[0, 1 - \dfrac{1}{2^n}\right] \end{array} \right\},$$

and

$$K(y, n) = \left\{ (x, y, z) \in A_y \;\middle|\; \begin{array}{l} x = 0 \text{ if } n = 0, \\ x \in \left[1 - \dfrac{1}{2^{2n-1}}, 1 - \dfrac{1}{2^{2n}}\right] \text{ if } n \neq 0, \\ z \in \left(1 - \dfrac{1}{2^n}, 2\right] \end{array} \right\}.$$

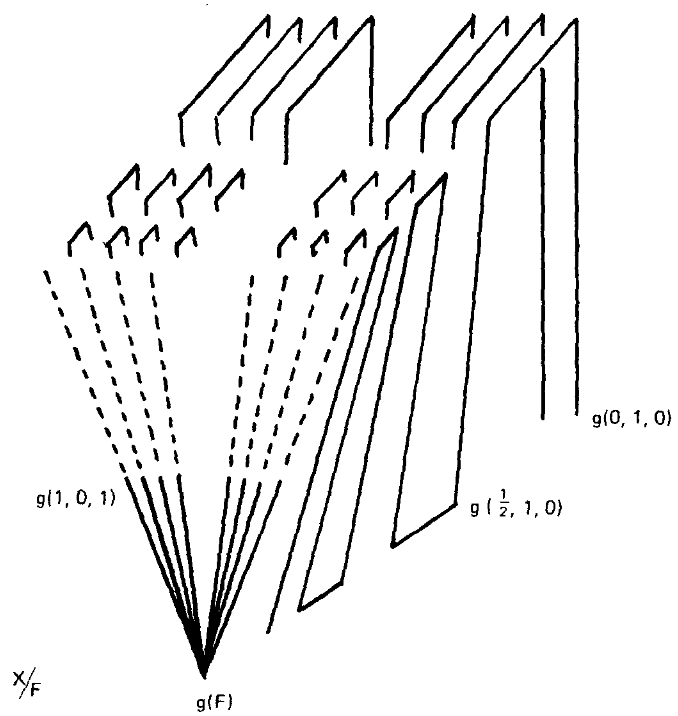

Fig. 6

Finally, we define $f: X/F \to X/F$ by the following rule:
For each (x, y, z) in X,

$$f(q(x,y,z)) = \begin{cases} q(1,0,1) & \text{if } x=1, \\ q(F) & \text{if } z=2, \\ q(1, (\frac{1}{3})^n y, 1) & \text{if } (x,y,z) \in H(y,n) \\ & \text{for some } n = 0,1,2,\ldots, \\ q\left(1, (\frac{1}{3})^n y, \dfrac{2-z}{1+(1/2^n)}\right) & \text{if } (x,y,z) \in K(y,n) \\ & \text{for some } n = 0,1,2,\ldots. \end{cases}$$

Then f is a continuous function and $f(q(F)) = q(1,0,1)$, i.e., $f(q(F)) \neq q(F)$ and the dispersion point is not fixed under f.

Remarks on Example 4.

(a) f is not a surjection, and $f(X/F)$ is homeomorphic to the Cantor teepee.

(b) Even though $q(F)$ is not a fixed point of f, f has $q(1,0,1)$ as a fixed point.

(c) X/F cannot be embedded in the plane.

The above remarks will lead us to the following questions.

Question 2. If X is a connected space with a dispersion point p, and if $f: X \to X$ a continuous surjection, then is $f(p) = p$?

Question 3. Does a dispersion point space have the fixed point property?

Question 4. If X is a connected space with a dispersion point p in the plane, and if $f: X \to X$ a nonconstant mapping, then is $f(p) = p$?

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